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ON A SET OF FOUR LINEAR ASSOCIATIVE ALGEBRAIC UNITS.

By S. A. COREY, Hiteman, Iowa.

Dr. Dickson has called attention in the November MONTHLY to a very remarkable set of non-associative algebraic units, and notes that among the many other similar (though usually associative) units, the best known are Hamilton's famous unit vectors. i, j, k , which on account of being perfectly isotropic when applied to Euclidean space, have been of much use to scientists in simplifying the language of analysis as applied to the physical sciences. In this connection it may be of interest to notice very briefly another somewhat remarkable set of "units," in terms of which, when interpreted in a certain manner, scalars, vectors, and quaternions may all alike be written.

We shall begin, as we may, by defining these "units" by a "multiplication table." This is virtually what Hamilton did when he defined his "units" in such a way as to admit of their bearing the interpretation he desired to place on them as unit vectors. Let, then, H, I, J, K , be these four "units" of which the following is the "multiplication table:"

	H	I	J	K
H	H	I	H	$-I$
I	H	I	$-H$	I
J	$J-K$	J	K	
K	$-J$	K	J	K

Further assume that multiplication of these "units" with ± 1 is always commutative, *e. g.* $H(-J) = -HJ$.

Then will multiplication always be associative; for, taking these "units" in sets of three, thus,

$$H.IJ = -HH = -H, \quad HI.J = IJ = -H, \quad J.HK = -JI = K, \quad JH.K = JK = K,$$

and so on until all possible permutations of the four "units" taken three at

a time have been enumerated. In every case multiplication will be found to be associative. Similarly taking the four "units" four at a time, thus,

$$HIJK=H.IJK=HIJ.K=H.I.JK=H.IJ.K=HI.JK=IK=I,$$

$$HJIK=H.JIK=HJI.K=H.J.IK=H.JI.K=HJ.IK=HI=I,$$

and so on, until all possible permutations of the four "units" taken four at a time have been enumerated. In every case multiplication will be found to be associative. But as any and all permutations or combinations of the four "units" taken five or more at a time can readily be reduced to some of the foregoing permutations of four letters, it follows that multiplication is *always* associative.

After thus defining our "units" we may impose further restrictions, provided only these restrictions in no way conflict with the restrictions imposed by our "multiplication table." By so doing we simply exclude from consideration all interpretations of these "units" which do not admit of these added restrictions. Let these added restrictions be as follows:

First. Our "units" may be added (and subtracted).

Second. Addition (and subtraction) must be associative, $\{[H+I]+J+K\}=[H+(I+J)+K]$, and so on.

Third. Addition (and subtraction) must be commutative, $[(H+I+J+K)=(J+H+K+I)]$, and so on.

Fourth. Multiplication must be distributive over addition, $[H(H+I+K+J)=(HH+HI+HK+HJ)]$, and so on.

Fifth. Multiplication, addition (and subtraction) of our "units" with ordinary scalars (rational and irrational) must be associative, commutative, and distributive, exactly as if these "units" were ordinary scalars.

It should be particularly noticed that no assumption has been made that division is always possible, nor that the law of indices holds. For this reason no operation on our "units" involving either division or the law of indices is permissible, until, in any particular case, these operations have been proven permissible.

We may now seek to interpret the meaning of these "units" in such a way as to satisfy all the imposed conditions or restrictions. With this purpose in view it may be noted that all the restrictions imposed, except those imposed by the "multiplication table" are the same as those that apply to algebraic numbers (scalars) and vectors. We know that vector multiplication is non-commutative. In view of these facts it may be worth while to try to so combine algebraic numbers and vectors as to form combinations the multiplication of which is in accordance with the above "multiplication table." For, should we succeed in forming such combinations, they must fulfill *all* the requirements imposed on our "units," and, therefore, may be treated as *particular* values of these "units." No great difficulty is involved in finding a number of combinations which do not change in value when

squared (none of our "units" change in value when squared), and but little ingenuity is required to combine a set of four of them in such a way as to fulfill all the stated requirements. One such set is

$$\left. \begin{aligned}
 H &= \frac{1}{2}[1 - j - \theta(i + k)] \\
 I &= \frac{1}{2}[1 + j + \theta(i - k)] \\
 J &= \frac{1}{2}[1 + j - \theta(i - k)] \\
 K &= \frac{1}{2}[1 - j + \theta(i + k)] \text{ whence} \\
 1 &= \frac{1}{2}(H + I + J + K) \\
 i &= \frac{\theta}{2}(H - I + J - K) \\
 j &= \frac{1}{2}(-H + I + J - K) \\
 k &= \frac{\theta}{2}(H + I - J - K)
 \end{aligned} \right\} \quad (A)$$

where $\theta = \pm 1$, and i, j , and k are Hamilton's unit vectors. Other similar combinations can readily be obtained from this set by a cyclical permutation of the unit vectors, i, j, k , or by substituting for these vectors other rectangular unit vectors. All such combinations must, however, be considered as mere variations of (A). We know that in ordinary algebra we may substitute such operators as $d/dx, d/dy, d/dz$, etc., for ordinary algebraic numbers because their laws of combination are identical. Whether, likewise, such distinct values of H, I, J , and K exist is a matter of speculation, but seems not impossible.

It might have been a more logical and simple process to have commenced by assuming (A) and then deducing the "multiplication table" therefrom. Such a method of procedure would have led up to the same results, but would not have made it so evident that the values of the fundamental "units" given in (A) are mere *particular* values.

By closely examining (A) we learn that this interpretation of our "units" implies the following:

First. Each "unit" is a Cayleyan nullitat (quaternion with zero norm).

Second. All quaternions, including scalars and vectors, can be written in terms of H, I, J , and K , and vice versa.

Third. These "units" may be algebraically combined in accordance with the same laws that govern the combination of quaternions. (If division and the law of indices be excepted, there seems to be no reason why the laws of quaternions should not govern in the combination of other entirely distinct interpretations of these "units," should such interpretations be found.)

Let $Q = (aH + bI + cJ + dK)$, $R = (eH + fI + gJ + hK)$, $W = (rH + sI + tJ + uK)$, $a, b, c, d, e, f, g, h, r, s, t$, and u , being numbers in the algebraic

field, operators, such as d/dx , d/dy , d/dz , etc., or a combination of such numbers and operators. Then, if $Q.R=W$,

$$\begin{aligned} r &= e(a+c) + f(a-c), \quad s = e(b-d) + f(b+d), \\ t &= g(c+a) + h(c-a), \quad \text{and } u = g(d-b) + h(d+b). \end{aligned}$$

Using ordinary quaternion methods and Hamilton's notation,

$$\begin{aligned} SQ &= \frac{1}{2}(a+b+c+d), \\ VQ &= (Q-SQ) = (a-w)H + (b-w)I + (c-w)J + (d-w)K, \\ &\quad [w \equiv \frac{1}{4}(a+b+c+d)], \\ TQ &= \sqrt{[2(ad+bc)]}, \quad \text{and} \\ UQ &= (2ad+2bc)^{-\frac{1}{2}}(aH+bI+cJ+dK). \end{aligned}$$

Similar expressions can readily be found for SUQ , VUQ , TVQ , $TVUQ$, etc. It would, indeed, be possible to construct an entire system of quaternions in which H , I , J , K would be the "units" used to replace unity and the i , j , k of Hamilton. Such a system would necessitate the use of irrational numbers in expressions involving ordinary vectors, and would, therefore, not be well suited for use in problems which concern the physicist; it would, however, have the remarkable characteristic that scalars, vectors, and vector products could all alike be expressed in terms of the same linear, homogeneous units, although such expressions would be cumbersome and of little practical value to the physicist. It is a matter of historic interest that Hamilton was much concerned about the heterogeneous character of his vector products, and tried to find an "*extra-spacial*" unit which would render such products homogeneous.* The $\sqrt{-1}$ of algebra may, perhaps, not be considered such an "*extra-spacial*" unit, but may evidently be used to obtain the result Hamilton sought to obtain by the use of such a unit. Inasmuch as the heterogeneity of vector multiplication disappears when $\sqrt{-1}$ is introduced by the linear substitutions involved in these "units," we may, perhaps, better conclude that the heterogeneity which Hamilton sought to remove was, after all, only a seeming heterogeneity resulting from the point of view afforded by his particular system of fundamental unit vectors, whereas quaternion analysis as a system is, of course, independent of any particular set of fundamental unit vectors.

But, in conclusion, we may observe that the algebraic properties of these "units" had been defined and proven consistent before any attempt was made to give them an interpretation. Had we, indeed, entirely failed to find an interpretation, these "units" would, nevertheless, have been realities, in an abstract sense, to the pure algebraist. The fact that they happen to be isomorphic to a certain set of four imaginary quaternions is, of course, not without interest to him, but should be looked upon by him as a mere coincidence.

*See article on Quaternions by Professor Tait in *Encyclopedia Britannica*.